

THE ADDITION FORMULA FOR LITTLE q -LEGENDRE POLYNOMIALS AND THE $SU(2)$ QUANTUM GROUP*

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Abstract. From the interpretation of little q -Jacobi polynomials as matrix elements of the irreducible unitary representations of the $SU(2)$ quantum group an addition formula is derived for the little q -Legendre polynomials. It involves an expansion in terms of Wall polynomials. A product formula for little q -Legendre polynomials follows by q -integration.

Key words. quantum groups, $SU(2)$, little q -Jacobi polynomials, little q -Legendre polynomials, Wall polynomials, addition formula, product formula

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1. Introduction. Quantum groups were recently introduced by Drinfeld [5] and Woronowicz [17]. Interesting examples are provided by deformation into a noncommutative Hopf algebra of some suitable commutative Hopf algebra of functions on a specific group. The most elementary nontrivial example comes from deformation of the algebra of polynomials on $SU(2)$ (cf. Woronowicz [18]). We denote the resulting quantum group by $SU_\mu(2)$.

It has been proved by Vaksman and Soibelman [15], Masuda et al. [11], [12], and Koornwinder [10] that the matrix elements of the irreducible unitary representations of the quantum group $SU_\mu(2)$ can be expressed in terms of the little q -Jacobi polynomials. In the present paper we use this interpretation to derive an addition formula for the little q -Legendre polynomials, i.e., for the q -analogues of the Legendre polynomials within the class of little q -Jacobi polynomials. The derivation of this formula is straightforward and analogous to a proof of the addition formula for Legendre polynomials using irreducible representations of $SU(2)$ (cf. Vilenkin [16, Chap. 3]). However, the resulting formula involves noncommuting variables. It is less easy to rewrite it equivalently as a formula involving only commuting variables. We can do this by using an infinite-dimensional irreducible $*$ -representation of the Hopf algebra considered as a $*$ -algebra. The result (Theorem 4.1) gives an expansion in terms of Wall polynomials (little q -Jacobi polynomials with the second parameter equal to zero).

Our addition formula somewhat resembles an addition formula for (continuous) q -ultraspherical polynomials derived by Rahman and Verma [14], but for that formula a (quantum) group-theoretic interpretation is not yet known. It would have been hard to find our formula without guidance from the quantum group. Indeed, that it is possible to obtain such a formula demonstrates the power and depth of the quantum group-theoretic interpretation of special functions. It turns out to be highly nontrivial to prove this formula analytically or to show that its limit case for $q \uparrow 1$ is the addition formula for Legendre polynomials. The analytic proof has been done by Rahman [13] and the limit result is proved by Van Assche and Koornwinder [2].

The contents of this paper are as follows. In §§ 2 and 3 the preliminaries about q -hypergeometric orthogonal polynomials and quantum groups, respectively, are presented. The derivation of the addition formula is given in § 4. Finally, a product formula for little q -Legendre polynomials is derived from the addition formula in § 5.

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2. Some q -hypergeometric orthogonal polynomials. Let $1 \neq q \in \mathbb{C}$. We use the familiar definitions and notation for q -shifted factorials and q -hypergeometric functions (cf. [6, Chap. 1], [10, § 2]). The *little q -Jacobi polynomials*

$$(2.1) \quad p_n(x; a, b|q) := {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx \right)$$

occur as part of a classification by Hahn [7] of orthogonal polynomials satisfying q -difference equations. Their detailed properties as orthogonal polynomials are given by Andrews and Askey [1]. For $a = b = 1$ we will call these polynomials *little q -Legendre polynomials*.

The special little q -Jacobi polynomials obtained by putting $b := 0$ in (2.1) are known as *Wall polynomials*; cf. Chihara [3, § 5, Case I], [4, Chap. 6, § 11]:

$$(2.2) \quad p_n(x; a, 0|q) = {}_2\phi_1(q^{-n}, 0; aq; q, qx)$$

(Chihara uses another notation). We will put $a := q^\alpha$. These polynomials can be viewed as one of the many q -analogues of the Laguerre polynomials in view of the limit formula

$$\lim_{q \uparrow 1} p_n((1-q)x; q^\alpha, 0|q) = L_n^\alpha(x)/L_n^\alpha(0).$$

By specialization of the orthogonality relations for the little q -Jacobi polynomials we obtain the orthogonality relations for the Wall polynomials:

$$(2.3) \quad \frac{(q^{\alpha+1}; q)_\infty}{(1-q)(q; q)_\infty} \int_0^1 p_n(t; q^\alpha, 0|q) p_m(t; q^\alpha, 0|q) t^\alpha (qt; q)_\infty d_q t = \delta_{n,m} \frac{q^{n(\alpha+1)}(q; q)_n}{(q^{\alpha+1}; q)_n},$$

where the q -integral is defined by

$$\int_0^1 f(t) d_q t := \sum_{k=0}^{\infty} f(q^k)(q^k - q^{k+1}),$$

and where we suppose that $0 < q < 1$ and $\alpha > -1$. From [3, Forms. (5.1), (5.2)] together with (2.2) we obtain the three-term recurrence relation

$$(2.4) \quad \begin{aligned} xp_n(x; a, 0|q) &= -q^n(1 - aq^{n+1})p_{n+1}(x; a, 0|q) \\ &\quad + q^n(1 + a - aq^n - aq^{n+1})p_n(x; a, 0|q) \\ &\quad - q^n(a - aq^n)p_{n-1}(x; a, 0|q), \\ p_{-1}(x; a, 0|q) &= 0, \quad p_0(x; a, 0|q) = 1. \end{aligned}$$

Put

$$(2.5) \quad \begin{aligned} P_n(q^k; q^\alpha|q) &:= \left(\frac{(q^{\alpha+1}; q)_\infty (q^{k+1}; q)_\infty (q^{\alpha+1}; q)_n}{(q; q)_\infty (q; q)_n} \right)^{1/2} \\ &\quad \cdot (-1)^n q^{(k-n)(\alpha+1)/2} p_n(q^k; q^\alpha, 0|q). \end{aligned}$$

Then (2.3) can be rewritten as

$$(2.6) \quad \sum_{k=0}^{\infty} P_n(q^k; q^\alpha|q) P_m(q^k; q^\alpha|q) = \delta_{n,m}, \quad n, m = 0, 1, 2, \dots$$

Since the orthogonality measure in (2.3) has compact support, the orthonormal system

$$\{P_n(q^k; q^\alpha|q)\}_{k=0,1,2,\dots}, \quad n = 0, 1, 2, \dots,$$

is complete in the Hilbert space l^2 . Hence, we have also the dual orthogonality relations

$$(2.7) \quad \sum_{n=0}^{\infty} P_n(q^k; q^\alpha|q) P_n(q^l; q^\alpha|q) = \delta_{k,l}, \quad k, l = 0, 1, 2, \dots$$

We conclude this section with an expression of Wall polynomials in terms of a ${}_3\phi_2$:

$$(2.8) \quad p_n(x; q^\alpha, 0|q) = \frac{(-1)^n q^{n(n+2\alpha+1)/2} x^n}{(q^{\alpha+1}; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-n-\alpha}, x^{-1} \\ 0, 0 \end{matrix}; q, q \right).$$

This follows by putting $b := 0$ in

$$\begin{aligned} {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx \right) &= (q^{-n+1}x; q)_n {}_2\phi_2 \left(\begin{matrix} q^{-n}, q^{-n}b^{-1} \\ qa, q^{-n+1}x \end{matrix}; q, q^{n+2}abx \right) \\ &= \frac{(qb; q)_n q^{n(n-1)/2} (-aqx)^n}{(qa; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-n}\alpha^{-1}, x^{-1} \\ qb, 0 \end{matrix}; q, q \right). \end{aligned}$$

Here we have used a transformation formula for ${}_2\phi_1$ (cf. [6, Chap. 1]) in the first equality and reversion of summation order in the second equality (see also [8]).

3. The quantum group $SU_\mu(2)$. In the rest of this paper we fix $0 \neq \mu \in (-1, 1)$. Let \mathcal{A} be the unital $*$ -algebra generated by the two elements α and γ satisfying the relations

$$\begin{aligned} \alpha\gamma &= \mu\gamma\alpha, & \alpha\gamma^* &= \mu\gamma^*\alpha, & \gamma\gamma^* &= \gamma^*\gamma, \\ \alpha^*\alpha + \gamma\gamma^* &= I, & \alpha\alpha^* + \mu^2\gamma\gamma^* &= I. \end{aligned}$$

Let $\Phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be the unital $*$ -homomorphism such that

$$\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \quad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Then Φ acts as a comultiplication and \mathcal{A} thus becomes a Hopf algebra with involution which we say to be *associated with the compact matrix quantum group $SU_\mu(2)$* . In the limit for $\mu \uparrow 1$, \mathcal{A} becomes the algebra of polynomials in the matrix elements of the natural representation of $SU(2)$, and the comultiplication is then induced by the group structure of $SU(2)$.

It is possible to embed the $*$ -algebra \mathcal{A} as a dense $*$ -subalgebra of a C^* -algebra by a universal construction. The C^* -algebra approach is emphasized in particular by Woronowicz [17], [18]. In this paper we could work with the C^* -algebra, but only elements of the dense $*$ -subalgebra \mathcal{A} will be needed, so we will use this latter algebra. Other references for $SU_\mu(2)$ besides [17] and [18], are [5], [15], [11], and [12].

The irreducible unitary co-representations of \mathcal{A} (which are called irreducible unitary representations of $SU_\mu(2)$ in [18] and [10]) have been completely classified [18], [9], [15], [11], [12], [10]. Up to equivalence, there is one such co-representation for each finite dimension. We will denote the co-representation of dimension $2l+1$ by $t^{l,\mu}$ ($l=0, \frac{1}{2}, 1, \dots$), and its matrix elements with respect to a suitable orthonormal basis corresponding to the quantum subgroup $U(1)$ by $t_{n,m}^{l,\mu}$ ($n, m = -l, -l+1, \dots, l$). Then the co-representation property of $t^{l,\mu}$ is expressed by

$$(3.1) \quad \Phi(t_{n,m}^{l,\mu}) = \sum_{k=-l}^l t_{n,k}^{l,\mu} \otimes t_{k,m}^{l,\mu}.$$

The $t_{n,m}^{l,\mu}$ have been computed explicitly in terms of little q -Jacobi polynomials (cf. [15], [11], [12], [10]). Here we will only need the cases that $l=0, 1, 2, \dots$ and m or $n=0$. Put

$$(3.2) \quad p_{l,k}^\mu(x) := \left[\begin{matrix} l \\ k \end{matrix} \right]_{\mu^2}^{1/2} \left[\begin{matrix} l+k \\ k \end{matrix} \right]_{\mu^2}^{1/2} \mu^{-k(l-k)} p_{l-k}(x; \mu^{2k}, \mu^{2k} | \mu^2),$$

$$(3.3) \quad p_l^\mu(x) := p_{l,0}^\mu(x) = p_l(x; 1, 1 | \mu^2).$$

Here the notation (2.1) for the little q -Jacobi polynomials is used and (3.3) gives a little q -Legendre polynomial. Then (cf. [10, Thm. 5.3]), for $k=0, 1, \dots, l$,

$$\begin{aligned}
 t_{k,0}^{l,\mu} &= (\alpha^*)^k p_{l,k}^\mu(\gamma\gamma^*) \gamma^k, \\
 t_{0,k}^{l,\mu} &= (\alpha^*)^k p_{l,k}^\mu(\gamma\gamma^*) (-\mu\gamma^*)^k, \\
 t_{-k,0}^{l,\mu} &= (-\mu\gamma^*)^k p_{l,k}^\mu(\gamma\gamma^*) \alpha^k, \\
 t_{0,-k}^{l,\mu} &= \gamma^k p_{l,k}^\mu(\gamma\gamma^*) \alpha^k.
 \end{aligned}
 \tag{3.4}$$

Hence

$$t_{0,0}^{l,\mu} = p_l^\mu(\gamma\gamma^*).
 \tag{3.5}$$

The irreducible $*$ -representations of the $*$ -algebra \mathcal{A} on a Hilbert space are classified in [15, Thm. 3.2]. There is a family of one-dimensional and a family of infinite-dimensional representations, both parametrized by the unit circle. We pick one of these infinite-dimensional representations: Let \mathcal{H} be a Hilbert space with orthonormal basis e_0, e_1, \dots . Put $e_{-1}, e_{-2}, \dots = 0$. We define a $*$ -representation of τ of \mathcal{A} on \mathcal{H} by specifying the action of the generators of \mathcal{A} :

$$\begin{aligned}
 \tau(\alpha)e_n &:= (1 - \mu^{2n})^{1/2} e_{n-1}, \\
 \tau(\alpha^*)e_n &:= (1 - \mu^{2n+2})^{1/2} e_{n+1}, \\
 \tau(\gamma)e_n &:= \mu^n e_n, \\
 \tau(\gamma^*)e_n &:= \mu^n e_n.
 \end{aligned}
 \tag{3.6}$$

4. Proof of the addition formula. Let $l=0, 1, 2, \dots$. A special case of (3.1) is

$$\Phi(t_{0,0}^{l,\mu}) = \sum_{k=-l}^l t_{0,k}^{l,\mu} \otimes t_{k,0}^{l,\mu}.$$

Hence, by (3.4) and (3.5),

$$\begin{aligned}
 p_l^\mu(\Phi(\gamma\gamma^*)) &= p_l^\mu(\gamma\gamma^*) \otimes p_l^\mu(\gamma\gamma^*) \\
 &+ \sum_{k=1}^l (\alpha^*)^k p_{l,k}^\mu(\gamma\gamma^*) (-\mu\gamma^*)^k \otimes (\alpha^*)^k p_{l,k}^\mu(\gamma\gamma^*) \gamma^k \\
 &+ \sum_{k=1}^l \gamma^k p_{l,k}^\mu(\gamma\gamma^*) \alpha^k \otimes (-\mu\gamma^*)^k p_{l,k}^\mu(\gamma\gamma^*) \alpha^k.
 \end{aligned}
 \tag{4.1}$$

This formula might already be called an addition formula for little q -Legendre polynomials p_l^μ . It involves noncommuting variables. In the limit, for $\mu \uparrow 1$, the variables commute and (4.1) becomes the classical addition formula for Legendre polynomials. In three steps we will rewrite (4.1) into a formula involving commuting variables: First, we represent (4.1) as an operator identity on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ by using the representation $\tau \otimes \tau$ (cf. (3.6)). Second, we let these operators act on the standard basis of $\mathcal{H} \otimes \mathcal{H}$. Thus we obtain a family of vector identities in $\mathcal{H} \otimes \mathcal{H}$. Third, we take inner products with respect to another suitable orthonormal basis of $\mathcal{H} \otimes \mathcal{H}$. This will yield a family of scalar identities.

Apply $\tau \otimes \tau$ to both sides of (4.1) and let both sides of the resulting operator equality act on $e_{x+y} \otimes e_y$. Then

$$\begin{aligned}
 p_l^\mu((\tau \otimes \tau)\Phi(\gamma\gamma^*)) e_{x+y} \otimes e_y &= p_l^\mu(\mu^{2x+2y})p_l^\mu(\mu^{2y}) e_{x+y} \otimes e_y \\
 &+ \sum_{k=1}^l (-1)^k \mu^{k(x+2y+1)} (\mu^{2(x+y+1)}; \mu^2)_k^{1/2} (\mu^{2(y+1)}; \mu^2)_k^{1/2} \\
 (4.2) \quad &\cdot p_{l,k}^\mu(\mu^{2x+2y})p_{l,k}^\mu(\mu^{2y}) e_{x+y+k} \otimes e_{y+k} \\
 &+ \sum_{k=1}^l (-1)^k \mu^{k(x+2y-2k+1)} (\mu^{2(x+y)}; \mu^{-2})_k^{1/2} (\mu^{2y}; \mu^{-2})_k^{1/2} \\
 &\cdot p_{l,k}^\mu(\mu^{2x+2y-2k})p_{l,k}^\mu(\mu^{2y-2k}) e_{x+y-k} \otimes e_{y-k}.
 \end{aligned}$$

(Remember the convention that $e_n = 0$ for $n < 0$.)

In order to say more about the left-hand side of (4.2) we consider the action of

$$\Phi(\gamma\gamma^*) = (\gamma \otimes \alpha + \alpha^* \otimes \gamma)(\gamma^* \otimes \alpha^* + \alpha \otimes \gamma^*)$$

on $e_{x+y} \otimes e_y$. We obtain

$$\begin{aligned}
 (\tau \otimes \tau)(\Phi(\gamma\gamma^*)) e_{x+y} \otimes e_y &= \mu^{x+2y+1} (1 - \mu^{2x+2y+2})^{1/2} (1 - \mu^{2y+2})^{1/2} e_{x+y+1} \otimes e_{y+1} \\
 &+ (\mu^{2x+2y} + \mu^{2y} - \mu^{2x+4y} - \mu^{2x+4y+2}) e_{x+y} \otimes e_y \\
 &+ \mu^{x+2y-1} (1 - \mu^{2x+2y})^{1/2} (1 - \mu^{2y})^{1/2} e_{x+y-1} \otimes e_{y-1}.
 \end{aligned}$$

Hence, if

$$f := \sum_{y=0}^{\infty} c_y e_{x+y} \otimes e_y$$

belongs to $\mathcal{H} \otimes \mathcal{H}$, then

$$\begin{aligned}
 (\tau \otimes \tau)(\Phi(\gamma\gamma^*))f &= \sum_{y=0}^{\infty} [c_{y-1} \mu^{x+2y-1} (1 - \mu^{2x+2y})^{1/2} (1 - \mu^{2y})^{1/2} \\
 (4.3) \quad &+ c_y (\mu^{2x+2y} + \mu^{2y} - \mu^{2x+4y} - \mu^{2x+4y+2}) \\
 &+ c_{y+1} \mu^{x+2y+1} (1 - \mu^{2x+2y+2})^{1/2} (1 - \mu^{2y+2})^{1/2}] e_{x+y} \otimes e_y.
 \end{aligned}$$

Now choose

$$c_y := P_y(\mu^{2z}; \mu^{2x} | \mu^2),$$

where P_y is defined in terms of Wall polynomials by (2.5). Then $f \in \mathcal{H} \otimes \mathcal{H}$ and, by (2.4), the expression in square brackets on the right-hand side of (4.3) is equal to $\mu^{2z} c_y$. Define

$$(4.4) \quad f_z^x := \sum_{y=0}^{\infty} P_y(\mu^{2z}; \mu^{2x} | \mu^2) e_{x+y} \otimes e_y.$$

Then, by the orthogonality relations (2.7), the vectors $\{f_z^x\}_{z=0,1,2,\dots}$ form an orthonormal basis of

$$(4.5) \quad \bigoplus_{y=0}^{\infty} \mathbb{C} e_{x+y} \otimes e_y.$$

We also have

$$(4.6) \quad (\tau \otimes \tau)(\Phi(\gamma\gamma^*))f_z^x = \mu^{2z} f_z^x.$$

Now take the inner product of both sides of (4.2) with respect to f_z^x and apply (4.4), (4.6), and the self-adjointness of $\Phi(\gamma\gamma^*)$ acting on $\mathcal{H} \otimes \mathcal{H}$. Assume also the convention that $P_n := 0$ and $p_n := 0$ for $n < 0$. Then we obtain

$$\begin{aligned}
 (4.7) \quad p_l^\mu(\mu^{2z})P_y(\mu^{2z}; \mu^{2x} | \mu^2) &= p_l^\mu(\mu^{2x+2y})p_l^\mu(\mu^{2y})P_y(\mu^{2z}; \mu^{2x} | \mu^2) \\
 &\quad + \sum_{k=1}^l (-1)^k \mu^{k(x+2y+1)} (\mu^{2(x+y+1)}; \mu^2)_k^{1/2} (\mu^{2(y+1)}; \mu^2)_k^{1/2} \\
 &\quad \cdot p_{l,k}^\mu(\mu^{2x+2y})p_{l,k}^\mu(\mu^{2y})P_{y+k}(\mu^{2z}; \mu^{2x} | \mu^2) \\
 &\quad + \sum_{k=1}^l (-1)^k \mu^{k(x+2y-2k+1)} (\mu^{2(x+y)}; \mu^{-2})_k^{1/2} (\mu^{2y}; \mu^{-2})_k^{1/2} \\
 &\quad \cdot p_{l,k}^\mu(\mu^{2x+2y-2k})p_{l,k}^\mu(\mu^{2y-2k})P_{y-k}(\mu^{2z}; \mu^{2x} | \mu^2).
 \end{aligned}$$

Finally substitute (3.2), (3.3), and (2.5) in (4.7) and replace μ^2 by q . Then we obtain the following theorem.

THEOREM 4.1 (addition formula for little q -Legendre polynomials). *For $x, y, z = 0, 1, 2, \dots$ we have*

$$\begin{aligned}
 (4.8) \quad &p_l(q^z; 1, 1 | q)p_y(q^z; q^x, 0 | q) \\
 &= p_l(q^{x+y}; 1, 1 | q)p_l(q^y; 1, 1 | q)p_y(q^z; q^x, 0 | q) \\
 &\quad + \sum_{k=1}^l \frac{(q; q)_{x+y+k}(q; q)_{l+k}q^{k(y-l+k)}}{(q; q)_{x+y}(q; q)_{l-k}(q; q)_k^2} \\
 &\quad \cdot p_{l-k}(q^{x+y}; q^k, q^k | q)p_{l-k}(q^y; q^k, q^k | q)p_{y+k}(q^z; q^x, 0 | q) \\
 &\quad + \sum_{k=1}^l \frac{(q; q)_y(q; q)_{l+k}q^{k(x+y-l+1)}}{(q; q)_{y-k}(q; q)_{l-k}(q; q)_k^2} \\
 &\quad \cdot p_{l-k}(q^{x+y-k}; q^k, q^k | q)p_{l-k}(q^{y-k}; q^k, q^k | q)p_{y-k}(q^z; q^x, 0 | q).
 \end{aligned}$$

Remark 4.1. We could have derived the final result (4.8) from (4.1) as well by using one of the other members of the series of infinite-dimensional irreducible $*$ -representations of \mathcal{A} as given by [15, Thm. 3.2]. The same result would also have been obtained by use of the faithful $*$ -representation of \mathcal{A} given in [18, Thm. 1.2]. Furthermore, it can be shown that formula (4.8) taken for all $x, y, z = 0, 1, 2, \dots$ and with $q = \mu^2$ is equivalent to (4.1).

Remark 4.2. It is possible to give a more conceptual interpretation of the occurrence of Wall polynomials in the addition formula. Namely, it can be shown that Wall polynomials have an interpretation as Clebsch–Gordan coefficients for the decomposition of the $*$ -representation $(\tau \otimes \tau) \circ \Phi$ of \mathcal{A} as a direct integral of irreducible $*$ -representations of \mathcal{A} .

5. The product formula for little q -Legendre polynomials. If we multiply both sides of (4.7) with $P_y(\mu^{2z}; \mu^{2x} | \mu^2)$ and sum over z , then by (2.6) we obtain the product formula

$$p_l^\mu(\mu^{2x+2y})p_l^\mu(\mu^{2y}) = \sum_{z=0}^{\infty} p_l^\mu(\mu^{2x})(P_y(\mu^{2z}; \mu^{2x} | \mu^2))^2.$$

After substituting (3.3), (2.5), and (2.8) and after replacing x by $x - y$ and μ^2 by q , we get the desired product formula.

THEOREM 5.1. For $x, y = 0, 1, 2, \dots$ we have

$$p_l(q^x; 1, 1|q)p_l(q^y; 1, 1|q) = (1-q) \sum_{z=0}^{\infty} p_l(q^z; 1, 1|q)K(q^x, q^y, q^z|q)q^z$$

with

$$K(q^x, q^y, q^z|q) := \frac{(q^{x+1}; q)_x (q^{y+1}; q)_x (q^{z+1}; q)_x q^{xy+xz+yz}}{(q; q)_x^2 (1-q)} \cdot \left\{ {}_3\phi_2 \left(\begin{matrix} q^{-x}, q^{-y}, q^{-z} \\ 0, 0 \end{matrix}; q, q \right) \right\}^2.$$

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